

# THE STRUCTURE SEMIOTIC ‘MYSTERY’ OF THE GRAPHS

John-Tagore Tevet

**Abstract:** Given a method, where by ‘semiotic invariants’ be constructed canonical presentation mode of the graphs can be solve various classical and non-classical problems in a compact non-classical way.

## Contents

SOME INITIAL OPINIONS	1
1. SEMIOTIC ATTRIBUTES AND THEIR USING	2
2. THE SYMMETRY PROBLEM: ORBITS	5
3. THE PROBLEMS OF CANONICAL REPRESENTATION AND ISOMORPHISM	6
4. THE PROBLEMS OF RECONSTRUCTIONS AND ADJACENT STRUCTURES	8
AN AFTERWORD	13

## SOME INITIAL OPINIONS

*Structure semiotics* [1], specify, *semiotics of the structure* is a research domain in the limitary of *graph theory* and *semiotics*. Under consideration is the *structure of a graph* as such.

*Structure* (Latin word *structura* (inner)building) is a *philosophical category*, that is defined as a *steady relation or organizing of system elements* [2]. Organizing is an integral phenomenon at connectivity, branching, regularities, symmetry etc. On the other side, structure is an abstraction of the system, its „skeleton“, where its elements and their relationships be lose at their empirical meanings and to substrate is only its organizing. For the concept of structure is inherent an singular, at the same time an universal relation type – *composition* – that is expressable mathematically by equations, matrices, graphs etc. The *qualitative differences* between structural elements depend on their connectivity mode in structure – at their *positions* in the structure. And so, structure is a formation of mutually related elements that is presentable in *graph* form. To note, that the exact definition of structure give by *isomorphism* concept. Structure is the *canonical representation* of a *class the isomorphic graphs*.

*Semiotics* deal with study the *meaning, communication- and interpretation processes*. A *sign* in semiotic system is an entity which signifies another entity, it is related with *cognition and thinking*. To structural signs are specific *semiotic invariants* of a graph, what have intrinsic meaning.

Certainly differs structure semiotic vision of the graphs in some respects at conventional. Emerge new truths about graphs. A most essential structural attribute is *symmetry*. Also there are various visions. Deeply is rooted the understanding about *symmetry* (**A**) as a kind of reflection. In fact is the symmetry concept more extend: *symmetry* (**B**) is a structural attribute that is expressed as a regular repetition (recurrence) of similar components (parts, particles) of an object in space and/or time [6]. The concept of symmetry (**B**) is *in mathematics* is defined as the existence the transitivity domain of automorphisms or *orbits* in *AutG*, i.e. in a graph *G*. An orbit is practically an *equivalence class*, whereof elements have the *equal positions* in structure *GS*. This conformity of positions be expressed as *symmetry* of structure.

Solution of the *reconstruction problem* takes place by *isomorphism classes* of greatest sub- and smallest super-graphs.

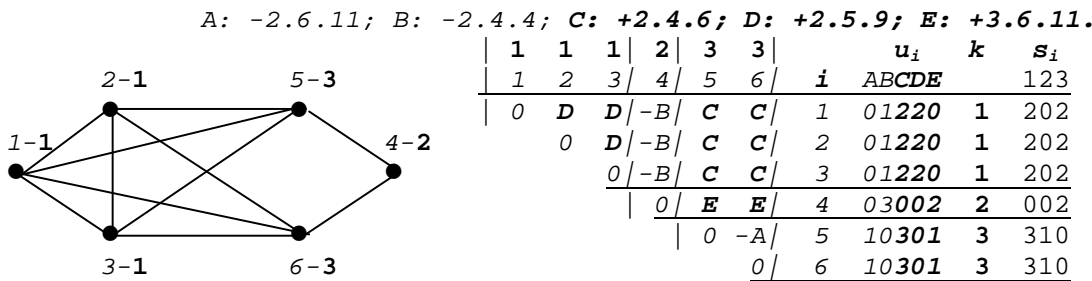
# 1. SEMIOTIC ATTRIBUTES AND THEIR USING

## 1.1. Semiotic invariants

We issue from guesswork, that structure of a graph is recognizable by identification its “elementary particles” i.e. by vertex pairs. A vertex pair  $v_i, v_j$  is identifiable by its “relation” in the form of a **pair graph**  $g_{ij}$  as a subgraph that represents the *intersection of their neighborhoods*.

Invariant  $\pm d.n.q_{.ij}$  of pair graph is a **semiotic invariant** of structure, where  $-d$  signify an usual distance between disadjacent vertices and  $+d$  a collateral distance between adjacent vertices  $v_i$  and  $v_j$ ;  $n$  – number of vertices and  $q$  – number of edges in the pair graph. Such semiotic invariant we call **pair sign**. A system of pair signs represented in the form of **sign matrix**  $S$ . It is realized on the ground of a Semiotic Identification Principle and is treatable as a *text* of graph structure. Structure is investigable just by a sign matrix.

**Example 1.1.** Results the Semiotic IP. Graph  $G$ , its pair signs, decomposed sign matrix  $S$  with frequency- and class vectors:



**Comment:** Sign matrix has recognized three vertex- and five vertex-pair classes, including three edge- and two “non-edge” classes and also the structure as a whole.

The pair signs identify the paths, girths, cliques etc attributes of a graph (Prop. 1.1). How to read the pair signs? What are their *structural meanings*? For example, a pair sign  $+dnq$ , where  $+d > 2$ , identify a **girth (cycle)** or their *assemblage* with length  $d+1$  and it is called **girth sign**. Etc.

## 1.2. Itemization the of regularities

Structure has various well-known and unknown *regularities*. These are readable from sign matrix  $S$ . There exist following regularities (Def. 1.2): 1) Graph, whereof vertices are with equal degree is **(degree)-regular**. 2) Graph, whereof all the distances between disadjacent vertex pairs are equal is **distance-regular**. 3) Graph, whereof all the  $n$  vertices belong the same number times to girths with length  $n-a$  is **girth-** or **circle-regular**. 4) Graph, whereof all the  $n$  vertices belong the same number times to clique with power  $n-b$  is **clique-regular**. 5) Graph said **strongly regular** with parameters  $(k,a,b)$  if it is a  $k$ -degree-regular incomplete connected graph such that any two adjacent vertices have exactly  $a \geq 0$  common neighbors and any two non-adjacent vertices have  $b \geq 1$  commons.

Both regularity and symmetry (**B**) constitute recurrence the “similar” elements. Wherein consist the distinction of regularity and symmetry? The “similarity” of regularity appears by the elementary properties, i.e., the properties of vertices and their pairs. The “similarity” of symmetry appears by the global properties, i.e., the properties that depend from the structure. *Similarity, as the equal positions* the vertices and vertex pairs in structure, i.e. orbits, are properties of the structure.

**Note:** The range numbers of definitions etc correspond there to the numbers in basic text.

### 1.3. Girth- and clique regularity

Clique recognition is a popular task of graph theory. Girth recognition for some reason or other not. Ideology of almost all clique recognition algorithms be grounded on the distinction the vertices that belong to a greatest clique, from others. Methods for that purpose are elaborated in gross. Unfortunately not go off to find a clique fan, who is interested on clique regularity.

For example (Ex. 1.2) *bisymmetric* and *strongly regular* Petersen graph is **5-girth regular**, it has 12 girths, where each vertex belong to six girths and each edge belong to four girths. Its complement is **4-clique regular**, it has 5 *intercrossed* partial cliques, where each vertex belongs to two cliques and each edge belong to one clique. If clique has a meaning and if it is findable then usually is it educe. In case of clique regularity so no done, even though touch with a warrantable phenomenon.

Some *relationships between symmetry, girth- and clique regularity* (Prop. 1.2): **a)** All the *vertex symmetric* (“transitive”) graphs are even if *girth regular* or *clique regular*, or *girth- and clique regular* – and contrariwise. **b)** The complement of a *bisymmetric* or of a *edge-symmetric girth regular* graph is *clique regular*. **c)** A *poly-symmetric* graph can be at the same time so *girth- and clique regular*. For example (Ex. 1.4), the complement of edge symmetric and 5-girth regular dodecahedron is **8-clique regular** with five *intercrossed* partial cliques.

*Partial cliques* (Prop. 1.3) of a clique regular graph can be *disconnected, mutually connected or intercrossed*. Complement of a *m-partite* graph in case of equal *n* parts is **n-clique regular**, with the number *m* of non-intercrossed *n-cliques*. For example (Ex. 1.5), complement of *bipartite* and 5-girth regular Heawood’s graph is **7-clique regular** with two *mutually connected* partial cliques.

### 1.4. Bisymmetry, clique- and strong regularity

There exist an interest relationship between bisymmetry, clique- and strong regularity. Bisymmetric graph has just one edge orbit and one “non-edge” orbit, it is represented only by two pair signs.

**Example 1.7.** Graph **B6-3**, its complement **B6-12**, their pair signs, sign matrices and common symmetry characteristics:

$A: -0.2.0; B: +1.2.1.$

1	2	3	4	5	6	<i>i</i>	<i>AB</i>	<i>deg</i>
0	<i>B</i>	<i>-A</i>	<i>-A</i>	<i>-A</i>	<i>-A</i>	1	<i>41</i>	1
	0	<i>-A</i>	<i>-A</i>	<i>-A</i>	<i>-A</i>	2	<i>41</i>	1
		0	<i>B</i>	<i>-A</i>	<i>-A</i>	3	<i>41</i>	1
			0	<i>-A</i>	<i>-A</i>	4	<i>41</i>	1
				0	<i>B</i>	5	<i>41</i>	1
					0	6	<i>41</i>	1

$A: -2.6.12; B: +2.4.5.$

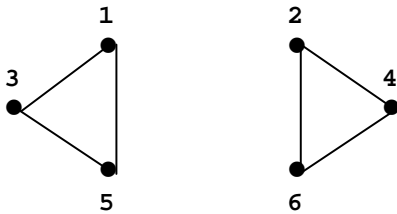
1	2	3	4	5	6	<i>i</i>	<i>AB</i>	<i>deg</i>
0	<i>-A</i>	<i>B</i>	<i>B</i>	<i>B</i>	<i>B</i>	1	<i>14</i>	4
	0	<i>B</i>	<i>B</i>	<i>B</i>	<i>B</i>	2	<i>14</i>	4
		0	<i>-A</i>	<i>B</i>	<i>B</i>	3	<i>14</i>	4
			0	<i>B</i>	<i>B</i>	4	<i>14</i>	4
				0	<i>-A</i>	5	<i>14</i>	4
					0	6	<i>14</i>	4

<i>SRV</i>	<i>HR</i>	<i>SR</i>	<i>aut</i>
$3^1 12^1$	$0.2173$	$0.8152$	$48$

Comments a) Graph **B6-3** and its complement **B6-12** are *bisymmetric*. b) Graph **B6-3** consist of *three disconnected 2-cliques*, it is *2-clique regular*. c) Complement **B6-12** is *three partite*, where its parts correspond to 2-cliques of **B6-3**. It is a so called *partite clique*, exactly with a *2-tri-clique*, generally called *n-m-clique*. It is simply sight that all the vertices belong to *triangles*.

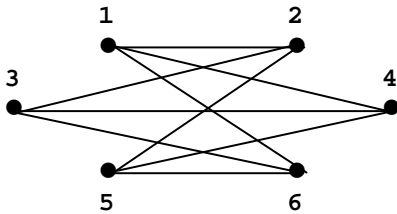
Example 1.8. Graph **B6-6**, its complement **B6-9**, their pair signs, sign matrices and common symmetry characteristics:

$$A: -0.2.0; B: +2.3.3.$$



	1	2	3	4	5	6	<i>i</i>	<i>ABC</i>	<i>deg</i>
0	-A	B	-A	B	-A		1	32	2
		0	-A	B	-A	B	2	32	2
			0	-A	B	-A	3	32	2
				0	-A	B	4	32	2
					0	-A	5	32	2
						0	6	32	2

$$A: -2.5.6; B: +3.6.9.$$



	1	2	3	4	5	6	<i>i</i>	<i>AB</i>	<i>deg</i>
0	B	-A	B	-A	B		1	23	3
		0	B	-A	B	-A	2	23	3
			0	B	-A	B	3	23	3
				0	B	-A	4	23	3
					0	B	5	23	3
						0	6	23	3

<i>SRV</i>	<i>HR</i>	<i>SR</i>	<i>aut</i>
$6^2 9^1$	0.2923	0.7515	72

Comments a) Graph **B6-6** and its complement **B6-9** are *bisymmetric*. b) Graph **B6-6** consist of *two disconnected 3-cliques*, it is *3-clique regular*. c) Complement **B6-9** is *bipartite*, where its parts correspond to 3-cliques of **B6-6**. d) **B6-9** is a *3-bi-clique* that is *4-girth regular*. e) Pair sign +3.6.9 cover all the  $n=6$  vertices and all the  $q=9$  edges, it is the *complete invariant* of **B6-9**.

On the *relationships between bisymmetry and partial cliques* (Prop. 1.4): a) The *complement* of a graph with  $m$  equal *disconnected cliques* is a *bisymmetric m-partite complete graph*, i.e. it is a *n-m-clique* – and contrariwise. For example, the complement of the structure with two disconnected cliques is a *bi-clique*, with three disconnected cliques is a *tri-clique*, etc. b) On the aspect of bisymmetry no essential is the structure *disconnected* or *connected*. b) All the *n-m-cliques* with equal power  $n$  and their complements are *bisymmetric*. c) A *n-m-clique* contain an usual clique with the power  $m$ , it is *m-clique regular*. For example, bi-clique is *2-clique regular*, tri-clique is *3-clique regular*, etc. d) A bisymmetric *n-m-clique* contain  $s=n^m$  usual cliques with power  $m$ . e) Correspondingly to the number of parts we call a *n-m-clique* to *bi-, tri-, quadro-, quinta-, sexta-, septa-, octa-, nona-, deca-, or undeca-* etc *-clique*. For example, there exist four *n-m-cliques* with 12 vertices – *2-sexta-, 3-quadro-, 4-tri-* and *6-bi-clique* as complements of disconnected 2-, 3-, 4- and 6-cliques correspondingly. Thus, the *n-m-cliques* are simply constructable.

The properties that are connected with *strong regularity* (Prop. 1.5): a) All the *connected bisymmetric 2-distance regular structures* are *strongly regular*, but no on the contrary. b) All the *n-m-cliques* are *strongly regular*, but no on the contrary. c) The *connected complement* of a strongly regular structure is also *strongly regular*.

Among the graphs with 20 vertices are 27 bisymmetric strongly regular *n-m-cliques* plus 12 bisymmetric strongly regular graphs that no are *n-m-cliques*. Consequently, among the structures with 1 to 20 vertices there exists 27+12=39 bisymmetric+strongly regular+clique- or girth regular graphs (Ex. 1.9).

Usually are the *lists of strongly regular graphs* incomplete. For example, in the list [9] lack 31 bisymmetric+strongly regular structures with 4 to 20 vertices. In the “most complete” list strongly regular structures [10] where are give 33 structures, among these also *n-m-cliques* lack unfortunately six structures. Semiotic approach discovers some new strongly- and clique regular structures.

## 2. THE SYMMETRY PROBLEM: ORBITS

### 2.1. Orbits: equal positions in the structure

*Symmetry* is there treated on its **B**-meaning, that expressed as existence *elements of the same kind*, specifically elements on the *equal positions* in structure that appears in the *equal remain structures* that obtained after removal the equal elements (Prop 2.1). Symmetry is a more essential attribute of structure.

Structural recognition of equal positions or symmetry take place by orbits in sign matrix **W**. Confuse term *orbit* is a concept of group theory and need here clarification. A permutation the vertices or vertex pairs that retain the structure called *automorphism  $\alpha$* . It is be expressed as an *inner- or local isomorphism (isomorphism with itself)*. Automorphisms form an automorphism group of graph *AutG*. The group divides to certain *transitivity domains* or *orbits  $\Omega$* , whereof elements take for *the same kind* or to *equivalent*. Usually be interested mainly on vertex orbits. Useful to begin at orbits of vertex pairs or *pair orbits*.

The relationships between *automorphisms, local isomorphisms, pair signs* and *pair orbits* are seated by corresponding principles (Prop. 2.2). A transitivity domain of *automorphisms* correspond to a *class the local isomorphisms  $\Gamma_n$*  of *remain structures* that are replaceable with *pair graphs*. The elements. i.e. pairs, of isomorphism class  $\Gamma_n$  are identified by equal *pair signs* in decomposed sign matrix and constitute a *pair orbit  $\Omega R_n$*  of edges or “non-edges”. The classes of vertices and vertex pairs on the Example 1.1 are in fact vertex-  $\Omega V_k$  and pair orbits  $\Omega R_n$  correspondingly. The orbits, that be recognized by group theoretic and structural ways, *coincide!* Graphs with different structures can be have one and same group *AutG*, but have different sign matrices **S**. Etc.

### 2.2. Symmetry kinds and measuring

There exist *symmetry kinds* of graph structure (Prop. 2.4), such as: **a)** a graph **G**, that has only one vertex orbit  $\Omega V_k$  is a *vertex symmetric graph* whatever also *transitive* called; **b)** vertex symmetric graph that has only one edge orbit (i.e. pair(+)-orbit)  $\Omega R_n^+$  and only one “non-edge” orbit (i.e. pair(-)-orbit)  $\Omega R_n^-$  is a *bisymmetric graph* (for example, Petersen graph); **c)** vertex symmetric graph that has one pair(+)-orbit  $\Omega R_n^+$  and several pair(-)-orbits  $\Omega R_n^-$  is an *edge symmetric* or (+)*symmetric graph* (for example Heawood’s graph); **d)** vertex symmetric graph that has several pair(+)-orbits  $\Omega R_n^+$  and several pair(-)-orbits  $\Omega R_n^-$  is a *poly-symmetric graph* (for example Franklin’s graph); **e)** a graph **G**, that has more than one vertex orbit  $\Omega V_k$ , whereby at least to one  $\Omega V_k$  belong at least two elements we call *partially symmetric graph* (for example Herschel’s graph).

Symmetry is *measurable* (Prop. 2.6, 2.7), its value is 1, if there exist only one orbit; the value is 0, if the number of orbits equal to the number of elements. It raise a possibility to *compare, order and grouping* the graphs with different size by symmetry values. But, how can be express or represent the orbits? It is very simply! A pair orbit  $\Omega R_n$  is a part of structure  $GS$  and consist also of vertices and edges. It is presentable as a partial graph.

### 2.3. Orbit structures

*Orbit structure* is a structure, whereof edges correspond to the certain pair signs that represent a certain pair orbit (Prop. 2.8). A structure, that correspond to an edge orbit (pair(+)-orbit) is a partial structure of initial structure, and this that correspond to an “non-edge” orbit (pair(-)-orbit) is a partial structure of the complement. These open the “hidden sides” of structure (Prop. 2.9), that perhaps also “mystical” can be. For example, an orbit structure of Folkman’s graph is Petersen graph, etc.

The main problem is there the *graph decomposition to its orbit-structures*. Orbit-graphs of vertex symmetric (bi-, mono- and polysymmetric) graphs cover all the vertices of the graph. Orbit-graphs of partially symmetric graphs cover only one vertex orbit.

Orbit-structures *open* its basic structure on the various aspects and *present its hidden properties* (Prop.2.10). If the basic structure is partite or contain components, cliques, girths etc, then emerge the corresponding vertex complexes in the orbit-structures in another form. It can be assert, that orbit-structures are “kindred” or “genetic derivatives” of their basic structure. Different orbit-structures of a graph or also different graphs *can be isomorphic or coincide*. For example, Hypercube and Möbius-Kantor graph have some common orbit-structures.

## 2. THE PROBLEMS OF CANONICAL REPRESENTATION AND ISOMORFISM

### 3.1. Canonical representation a graph

*Canonical representation a graph* means it show in a form which represents its *structure*, recommendatory *with exactness up to isomorphism*. This problem set up presumably Lazlo Babai [11, 12] in year 1977. Canonical representation constitute, for example, also the *3-cube codes* [13]. Unfortunately these no contain data about the structure self.

More deeply can be invade by semiotic invariants. Sign matrix  $S$  is a *canonical representation* of graph with exactness up to its *structure*, i.e. up to *pair signs, orbits* and *isomorphism* (Prop. 3.1). Thus, such canonical representation is more than isomorphism recognition. It is recognition of the structure completely, where essential role has orbit recognition. For example (Ex. 3.1), all the 15 *vertex orbits*, 80 *edge orbits* and 74 “*non-edge*” *orbits* of a *partially symmetric and strongly regular graph* are canonically presented on the ground of only *six pair signs*.

### 3.2. Deep identification the structure

In the case of large vertex symmetric graphs can be the pair signs  $\pm d.n.q_{ij}$  stay insufficient to for differentiation the pair orbits  $\Omega R_n$ . There we use the methods of *deep identification*. Using a *high degree m* pair graphs  $g_{ij}^m$  (P3.1.1), where to *complementary identifier* the pair sign  $dnq_{ij}^m$  of its high degree pair graph  $g_{ij}^m$ . Such deep identification we call *high identification*. Using a *local sign matrix*  $S_{ij}$  for identify the first or high degree pair graphs (P3.1.2). In such case take the *complementary identification* place on the ground local sign matrices  $S_{ij}$ . Such deep identification we call *local identification*. Using a *product*  $ExExEx...=E^n$  of an adjacency matrix, whereof

elements  $e_{ij}^n$  identify the *most length paths* between vertices (P3.1.3). Such deep identification we call *product identification*.

For example (Ex. 3.3) the *polysymmetric graphs*  $PRA_A$  and  $PRA_B$  have common pair signs:

$$A: -3.8.10; B: -3.6.7; C: -2.4.4; D: -2.3.2; \underline{E: +2.4.6}; F: +3.8.16.$$

There used the *product identification* method. Specified by matrix product  $E^{n=5}$  deep pair signs of graph  $PRA_A$ :

Marking the common pair signs	0	-A	-B	-C		-D	E		F
Productive pair signs $e^5$	180	125	110	165	160	80	231	233	210
Marking the deep pair signs	0	-A	-B	-C1	-C2	-D	E1	E2	F

Specified by matrix product  $E^{n=7}$  deep pair signs of graph  $PRA_B$ :

Common pair signs	0	-A	-B		-C			-D	E		F
Productive signs $e^7$	4410	3437	3276	3277	4081	4088	4011	3010	4831	4803	4445
Deep pair signs	0	-A	-B1	-B2	-C1	-C2	-C3	-D	E1	E2	F

Consequently, graphs  $PRA_A$  and  $PRA_B$  are *non isomorphic*

In case of hardly distinguishable bisymmetric and strongly regular graphs is suitable to use *local identification method*, where from non-isomorphism the pair graphs  $g_{ij}^A$  and  $g_{ij}^B$  of corresponding graphs  $G_A$  and  $G_B$  conclude *non-isomorphism* of  $G_A$  and  $G_B$  (Prop. 3.3, Ex. 3.5).

### 3.3. The graph isomorphism problem

The *isomorphism problem* is to design an algorithm that recognizes the isomorphism of two objects *The graph isomorphism problem* first came into prominence in 1857, when Arthur Cayley [16] reported his research on organic isomers.

*The classical isomorphism recognition* is an answer to the question, is graph  $G_A$  isomorphic to graph  $G_B$ ? If so, one must also provide the isomorphic substitution. On the structural aspect we can be say that graphs  $G_A$  and  $G_B$  are isomorphic *if and only if these have one and the same structure*.

Isomorphism problem is solvable also by comparison the sign matrices  $S_A$  and  $S_B$ .

**Example 3.8.** Decomposed sign matrices  $S_A$  and  $S_B$  of graphs  $G_A$  and  $G_B$ :

$$A: -2.5.7; B: +2.3.3; C: +2.4.6; D: +2.5.8; E: +3.6.11.$$

$\begin{array}{c cccc cc} 1 & 1 & 1 & 1 & 2 & 2 \\ \hline 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 0 & D & C & C & -A & B \\ & 0 & C & C & -A & B \\ & & 0 & D & B & -A \\ & & & 0 & B & -A \\ & & & & 0 & E \\ & & & & & 0 \end{array}$	$\begin{array}{c cccc cc} u_i & s_i & k & & & \\ \hline i & ABCDE & 12 & . & & \\ \hline 1 & 11210 & 31 & 1 & & \\ 2 & 11210 & 31 & 1 & & \\ 3 & 11210 & 31 & 1 & & \\ 4 & 11210 & 31 & 1 & & \\ 5 & 22001 & 21 & 2 & & \\ 6 & 22001 & 21 & 2 & & \end{array}$	$\begin{array}{c cccc cc} 1 & 1 & 1 & 1 & 2 & 2 \\ \hline 1 & 3 & 4 & 6 & 2 & 5 \\ \hline 0 & C & D & C & B & -A \\ & 0 & C & D & -A & B \\ & & 0 & C & B & -A \\ & & & 0 & -A & B \\ & & & & 0 & E \\ & & & & & 0 \end{array}$	$\begin{array}{c cccc cc} u_i & s_i & k & & & \\ \hline i & ABCDE & 12 & . & & \\ \hline 1 & 11210 & 31 & 1 & & \\ 2 & 11210 & 31 & 1 & & \\ 3 & 11210 & 31 & 1 & & \\ 4 & 11210 & 31 & 1 & & \\ 5 & 22001 & 21 & 2 & & \\ 6 & 22001 & 21 & 2 & & \end{array}$
---	--	---	--

Structural Equivalence Principle: **a)** Sign matrices of graphs  $G_A$  and  $G_B$  are *structurally equivalent*,  $S_A \approx S_B$ . Consequently, graphs are isomorphic  $G_A \cong G_B$ . **b)** Structural equivalence of sign matrices,

$S_A \approx S_B$  is *one-to-one correspondence* the partial matrices  $(W_{kk'})_A \leftrightarrow (W_{kk'})_B$ , pair orbits  $(\Omega R_n)_A \leftrightarrow (\Omega R_n)_B$  and their corresponding pair signs  $dnq_A \leftrightarrow dnq_B$ .

*Equivalence of structures*  $GS_A \equiv GS_B$  (Prop. 3.5) constitute an isomorphism  $G_A \equiv G_B$  with exactness up to components, parts, vertex orbits  $(\Omega V_k)_A \equiv (\Omega V_k)_B$ , pair orbits  $(\Omega R_n)_A \equiv (\Omega R_n)_B$ , orbit graphs  $(G_n)_A \equiv (G_n)_B$ , pair graphs  $(g_{ij})_A \equiv (g_{ij})_B$ , etc. Structural equivalence no needs isomorphism recognition on the aspect of vertex substitutions.

### **3.4. Canonical outputs of isomorphism algorithm**

Only few isomorphism recognition algorithms give a canonical output of processing results. Usually be limited laconically with phrase “isomorphic” or “not isomorphic”. Recognition the orbits no belong to isomorphism problem.

The correct *polynomial algorithm* of Ashay Dharvadker et al [8], (Ex.3.9) be grounded on the formation of incomplete sign matrices  $S_A$  ja  $S_B$ . These are decomposed up to frequency classes. In the framework of these classes take place rearrange the rows  $i$  and columns  $j$  to *isomorphism recognition with exactness up to substitutions*. The time complexity and recognition self are proved in detail. The results are excellently disained. On the classical view point it is the *general isomorphism recognition algorithm* with canonical and figurative outputs.

Isomorphism algorithm of Blazej Podsiadlo [33] has to canonical output of a graph present its *biggest value* that no contain data about the graph, but enable to differentiate these, better as for example 3-cube-codes. It do no realized up to substitutions (Ex. 3.10).

## **4. THE PROBLEMS OF ADJACENT STRUCTURES AND RECONSTRUCTION**

### **4.1. Relationships between isomorphic graphs and their $(G \setminus v_i)$ -subgraphs**

Here is suitable to begin with a theorem, proved by A. Titov in 1975 [35]: “If all the  $(G \setminus v_i)$ -subgraphs of graph  $G$  are isomorphic, then automorphism group  $AutG$  is transitive on the set of vertices  $V$ “. It mean that graph  $G$  is *vertex symmetric* (“transitive”), i.e. *there exists only one vertex orbit*  $\Omega V_{k=1=K}$  which correspond just to one isomorphism class  $\Gamma_{k=1=K}$  of  $(G \setminus v_i)$ -sub-graphs.

There we return anew to problems of the relationships between *equal positions* and *remain graphs* (Prop. 2.1) and relationships between *automorphisms*, *local isomorphisms*, *orbits* and *semiotic invariants* (Prop. 2.2). Indeed, vertex orbit  $\Omega V_{k=1=K} = \Omega(v_{i=1}, \dots, v_{i=|V|})_{k=1=K}$  is a *transitivity domain of automorphisms in  $AutG$*  that be expressed by an isomorphism class  $\Gamma_{k=1=K}$  of  $(G \setminus v_i)$ -sub-graphs and is represented in the form of an *equivalence class* of vertices in partial matrix  $S_k$  of decomposed sign matrix  $S$ .

Between a vertex orbit  $\Omega V_k$  and  $(G \setminus v_i)$ -subgraphs be valid following proposition (Prop. 4.1). “For each vertex orbit  $\Omega V_k = \Omega(v_{i1}, \dots, v_{iq})_k$ ,  $k \in [1, K]$ , correspond *an isomorphism class*  $\Gamma_k = \{(G \setminus v_i)_I \cong \dots \cong (G \setminus v_i)_q\}_k$ “. It is demonstrated on Examples 4.2 and 4.3. We state that an isomorphism class  $\Gamma_k$  is practically an “isomorphism clique”. i.e. all the pairs of  $(G \setminus v_i)$ -subgraphs are isomorphic.

From isomorphism of orbits ensue according to Prop. 4.1 next proposition (Prop. 4.2): “From orbit isomorphism,  $(\Omega V_k)_A \equiv (\Omega V_k)_B$ ,  $k \in [1, K]$ , of isomorphic graphs  $G_A$  and  $G_B$  ensue the *isomorphism of isomorphism classes*,  $(\Gamma_k)_A \equiv (\Gamma_k)_B$ ,  $k \in [1, K]$ , where  $(G \setminus v_i)_A \subset (\Gamma_k)_A$  and  $(G \setminus v_i)_B \subset (\Gamma_k)_B$ ”.

Isomorphism of isomorphism classes,  $(\Gamma_k)_A \cong (\Gamma_k)_B$ , of  $(G \setminus v_i)$ -sub-graphs, that accompany with isomorphism of vertex orbits,  $(\Omega V_k)_A \cong (\Omega V_k)_B$ , constitute practically *an union of isomorphism classes*  $(\Gamma_k)_A \cup (\Gamma_k)_B$ , where exist also inter-class isomorphisms  $(G \setminus v_i)_A \cong (G \setminus v_i)_B$ ,

According to Titov theorem a vertex symmetric graph has only one isomorphism class  $\Gamma_{k=1=K}$  of  $(G \setminus v_i)$ -sub-graphs, i.e. all its  $(G \setminus v_i)$ -sub-graphs are isomorphic.

#### 4.2. The adjacent structures: greatest sub- and smallest superstructures

There exist elementary operations with edges: 1) *removal an edge*  $G \setminus e_{ij}$  give a ***greatest sub-graph***  $G^{sub}$  of  $G$ ; 2) *addition an edge*  $G \cup e_{ij}$  give a ***smallest super-graph***  $G^{sup}$  of  $G$ . The greatest sub-graphs  $G^{sub}$  and smallest super-graphs  $G^{sup}$  both called ***adjacent graphs***  $G^{adj}$  of graph  $G$  (Def. 4.1).

There be valid following state (Prop. 4.3): “If the edge operations  $f$  have been applied ***disjunctively***,  $\{(f_{ij})_1 \vee \dots \vee (f_{ij})_q\}_n$  to the vertex pairs of an pair orbit  $\Omega R_n = \Omega(r_{ij1}, \dots, r_{ijq})_n$  of a graph  $G$ , then the disjunctive adjacent graphs of graph  $G$  form ***an isomorphism class***  $\Gamma_n = \{(G^{adj}_n)_1 \cong \dots \cong (G^{adj}_n)_q\}$ ”.

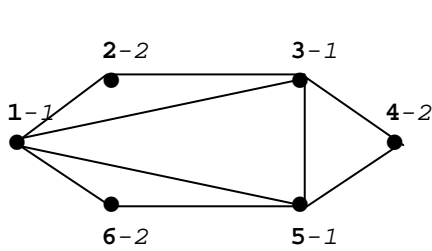
All the graphs  $G^{adj}$  of an isomorphism class  $\Gamma_n$  have the same structure and represent an ***adjacent structure***  $GS^{adj}$  of  $G$  is expressed by corresponding ***sign matrix***  $S^{adj}$ . In principle are all the graph structures  $GS$  also adjacent structures  $GS^{adj}_n$  of some structures.

An edge operation that disjunctively transforms structure  $GS$  to an adjacent structure  $GS^{adj}_n$  is called ***morphism***  $F_n$ ,  $F_n: GS \rightarrow GS^{adj}_n$  (Def. 4.2).

Therefore, morphism  $F_{n-}: GS \rightarrow GS^{sup}_{n-}$ , that is applied to a *pair(-)orbit*  $\Omega R_{n-}$  of  $GS$  induce an ***adjacent super-structure***  $GS^{sup}_{n-}$  of  $GS$  and morphism  $F_{n+}: GS \rightarrow GS^{sub}_{n+}$  that is applied to a *pair(+)-orbit*  $\Omega R_{n+}$  of  $GS$  induce an ***adjacent sub-structure***  $GS^{sub}_{n+}$  of  $GS$ .

**Example 4.7.** Graph structure  $GS.37$  (6.9.4) (by Graph Atlas [36] G163) and its *adjacent super- and sub-structures*:

$$A: -2.4.5; \quad B: -2.3.2; \quad C: +2.3.3; \quad D: +2.4.5.$$



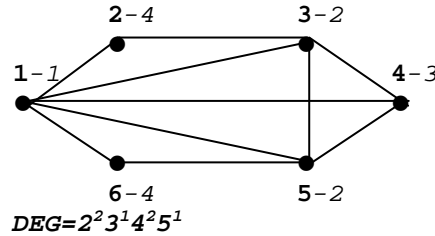
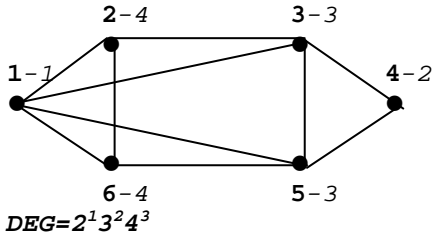
						$k$			
						$i$	$ABCD$		
1 1 1   2 2 2						1	1022	1	22
/ 1 3 5   2 4 6						3	1022	1	22
0 D D   C -A C						5	1022	1	22
0 D   C C -A						2	1220	2	20
0 / -A C C						4	1220	2	20
0 -B -B						6	1220	2	20
0 -B									
0									

Entry of identifiers of adjacent structures and characteristics of morphisms  $F_n$ :

$GS$	$GS^{adj}_n$	1	2
	$GS^{sup}_{n-}$	29	30
$GS.37$	$k.k'(p)$	2.2 (-B)	1.2 (-A)
	$PF^{sup}_{n-}$	3/6	3/6
	$GS^{sub}_{n+}$	72	76
$GS.37$	$k.k'(p)$	1.1 (+D)	1.2 (+C)
	$PF^{sub}_{n+}$	3/9	6/9

Where,  $GS^{sup}_{n-}$  and  $GS^{sub}_{n+}$  – range number adjacent structure correspondingly;  $k, k'$  – index of partial matrix  $S_{k, k'}$  that contain binary orbit, where  $(p)$  to concretize the pair orbit in therein;  $PF_n$  – percentage of pair orbit or *morphism probability*.

Adjacent super-structures by pair orbit  $-B$ ,  $GS^{sup}_{n=-B}$ , (GS.29 (6.10.11), by Graph Atlas G184) and by pair orbit  $-A$ ,  $GS^{sup}_{n=-A}$ , (GS.30 (6.10.12), by Graph Atlas G180) and their matrices  $S$ :



A: -2.5.8; B: -2.4.5; C: -2.3.2;  
D: +2.3.3; E: +2.4.5.

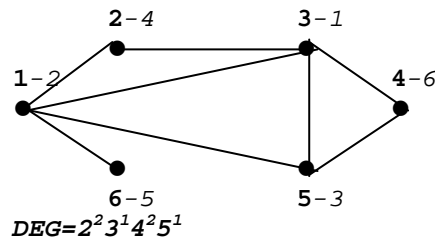
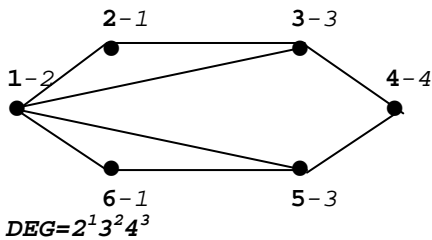
A: -2.4.5; B: -2.3.2;  
C: +2.3.3; D: +2.4.6; E: +2.5.8.

1/	2/	3	3/	4	4/	k			
1	4	3	5	2	6	i	ABCDE	1234	
0	-B	E	E	E	E	1	01004	1	0022
0	D	D	-C	-C	4	01220	2	0020	
0	E	D	-A	3	10022	3	1111		
0	-A	D	5	10022	3	1111			
0	D	2	10121	4	1011				
0	6	10121	4	1011					

1/	2	2/	3/	4	4/	k			
1	3	5	4	2	6	i	ABCDE	1234	
0	E	E	D	C	C	1	00212	1	0212
0	D	D	C	-A	3	10121	2	1111	
0	D	-A	C	5	10121	2	1111		
0	-A	-A	4	20030	3	1200			
0	-B	2	21200	4	1100				
0	6	21200	4	1100					

Comments: All the adjacent super-graphs by pair orbit  $-B$ ,  $GS^{sup}_{n=-B}$ , are *isomorphic* and all the adjacent super-graphs by pair orbit  $-A$ ,  $GS^{sup}_{n=-A}$ , are also *isomorphic*.

Adjacent sub-structures by pair orbit  $+D$ ,  $GS^{sub}_{n=+D}$ , (GS.72 (6.8.18), by Graph Atlas G148) and by pair orbit  $+C$ ,  $GS^{sub}_{n=+C}$ , (GS.76 (6.8.22), by Graph Atlas G137) and their matrices  $S$ :



A: -2.4.4; B: -2.3.2;  
C: +2.3.3; D: +3.4.4.

A: -3.5.6; B: -2.4.5; C: -2.3.2;  
D: +1.2.1; E: +2.3.3; F: +2.4.5.

1	1/	2/	3	3/	4/	k			
2	6	1	3	5	4	i	ABCD	1234	
0	-B	C	C	-B	-B	2	0320	1	0110
0	C	-B	C	-B	6	0320	1	0110	
0	C	C	-A	1	1040	2	2020		
0	-A	D	3	1121	3	1101			
0	D	5	1121	3	1101				
0	4	1202	4	0020					

1/	2/	3/	4/	5/	6/	k					
3	1	5	2	6	4	i	ABCDEF	123456			
0	F	F	E	-C	E	3	001022	1	011101		
0	E	E	D	-B	1	010121	2	101110			
0	-B	-C	E	5	011021	3	110001				
0	-C	-C	2	012020	4	110000					
0	-A	6	103100	5	010000						
0	4	111020	6	101000							

Comment: All the adjacent sub-graphs by pair orbit  $+D$ ,  $GS^{sub}_{n=+D}$ , are *isomorphic* and all the adjacent sub-graphs by pair orbit  $+C$ ,  $GS^{sub}_{n=+C}$ , are also *isomorphic*.

Let there exist the isomorphic graphs  $G_A$  and  $G_B$ . Then, in the case of each pair of pair orbits  $(\Omega R_n)_A$  and  $(\Omega R_n)_B$ ,  $n \in [1, N]$ , according to Prop. 4.2 exist isomorphic isomorphism classes,  $(\Gamma_n)_A \cong (\Gamma_n)_B$ ,  $n \in [1, N]$ , of corresponding adjacent graphs  $(G^{adj}_n)_A$  and  $(G^{adj}_n)_B$ , where  $(G^{adj}_n)_A \subset (\Gamma_n)_A$  and  $(G^{adj}_n)_B \subset (\Gamma_n)_B$ . We already know, that isomorphic graphs  $G_A \cong G_B \cong G_C \cong \dots$  have isomorphic adjacent graphs  $(G^{adj}_n)_A \cong (G^{adj}_n)_B \cong (G^{adj}_n)_C \cong \dots$ .

Now we can to formulate an essential Proposition:

**Proposition 4.4.** If the morphisms  $F_n: GS \rightarrow GS^{adj}_n$  have been disjunctively applied  $F_1 \vee \dots \vee F_n \vee \dots \vee F_N$  to the orbits  $\Omega R_1, \dots, \Omega R_n, \dots, \Omega R_N$  of a graph structure  $GS$ , then we say that structure  $GS$  is **deconstructed** to its adjacent structures  $\{GS^{adj}_n\} = GS^{adj}_1, \dots, GS^{adj}_n, \dots, GS^{adj}_N$ .

Indivisible structures cannot exist. We already know, that adjacent structure  $GS^{adj}_n$  mean an *isomorphism class*  $\Gamma_n^{adj}$  that can be contain isomorphic adjacent graphs  $(G^{adj}_n)_1 \cong (G^{adj}_n)_2 \cong (G^{adj}_n)_3 \cong \dots \subseteq GS^{adj}_n = \Gamma_n^{adj}$ .

### 4.3. On the Ulam's Conjecture

Reconstruction means anything restore. On the viewpoint of *Ulam's Conjecture* [7] mean it a relationship between two graphs and their  $(G \setminus v_i)$ -sub-graphs. We demonstrate that the graph reconstructions are in the structure semiotics aspect a definitely solvable problem by vertex orbits  $\Omega V_k$  and corresponding isomorphism classes of  $(G \setminus v_i)$ -sub-graphs. We demonstrate also that in addition to vertex reconstructions there exist also edge and "non-edge" reconstructions. It can be to construct a "constructive system of reconstructions".

We issue from this that graph  $G$  is represented in the canonical form, i.e. in the form of *complete sign matrix*  $S^*$  that represent  $G$  with exactness up to isomorphism and orbits. It mean that all the isomorphic graphs,  $\{G_1 \cong \dots \cong G_m \cong \dots \cong G_M\}_k$  and only these, form an *isomorphism class*  $\Gamma_k$  that is presented by corresponding *equivalent sign matrices*  $S^*_1 \approx \dots \approx S^*_m \approx \dots \approx S^*_M$

The classical treatment of reconstruction, the well-known *Ulam's Conjecture* is formulated as follows: "Let graph  $G$  has  $p$  vertices  $v_i$  and  $H$  has  $p$  vertices  $u_i$ , with  $p \geq 3$ . If for each  $i$ , the sub-graphs  $G_i = G \setminus v_i$  and  $H_i = H \setminus u_i$  are isomorphic, then the graphs  $G$  and  $H$  are isomorphic".

On structure semiotic aspect:

- Isomorphic graphs  $G$  and  $H$  constitute no more than *different graphs of one and the same isomorphism class*  $\Gamma$ ,  $G \& H \subset \Gamma$ , that have *one and the same structure*  $GS$  and orbits that are presented by equivalent sign matrices  $S_G^* \approx S_H^*$ .
- All the isomorphic sub-graphs  $G \setminus v_i$  and  $H \setminus u_i$  divide to common *sub-structures*  $(GS \setminus v_i)_1, \dots, (GS \setminus v_i)_k, \dots, (GS \setminus v_i)_K$  of isomorphic graphs  $G$  and  $H$ .

According to grand old man W.T.Tutte [37] the solution of reconstruction problem must start just from *isomorphism classes*.

Consequently, we ignore the wording of Ulam's Conjecture, but not its sense, i.e.: **1)** To study the relationship between isomorphic graphs and their  $(G \setminus v_i)$ - and  $(G \setminus e_{ij})$ -sub-graphs. **2)** At the level of "pair-isomorphism"  $G \cong H$  go to the level of *isomorphism classes*  $\Gamma$  or *-cliques*, i.e. to the level of structure  $GS$ .

#### 4.4. Reconstruction: the opposite operation of deconstruction

To reconstruction of a graph structure  $GS$  we take there reconstructability the structure by its adjacent structures  $GS^{adj}_n$ :

**Proposition 4.5.** If structure  $GS$  is *deconstructed* to its adjacent *substructures*  $GS^{sub}_1, \dots, GS^{sub}_m, \dots, GS^{sub}_N$ , i.e. to its greatest substructures  $(GS \setminus e_{ij})_n$ , then their *union*  $\cup (GS \setminus e_{ij})_n, n^+ \in [1, N^+]$ , *reconstruct* the same structure  $GS$ .

**Comments:** **a)** Let  $G$  is represented in the form of adjacent matrix  $E$ . Deconstruction the structure  $GS$  to its adjacent substructures mean by Prop. 4.4 strictly disjunctive removal an edge  $(E \setminus e_{ij})_1 \vee \dots \vee (E \setminus e_{ij})_n \vee \dots \vee (E \setminus e_{ij})_N$  in the case of each edge or pair(+)orbit  $\Omega R_n$ . This mean that in each adjacent matrix  $E_n$  lack a connection. Consequently their union reconstruct the graph,  $(E \setminus e_{ij})_1 \cup \dots \cup (E \setminus e_{ij})_n \cup \dots \cup (E \setminus e_{ij})_N = E, E \equiv G$ . **b)** Principally be valid (a) also then, when graph is presented by its list  $L$  of adjacent vertices  $(L \setminus e_{ij})_1 \cup \dots \cup (L \setminus e_{ij})_n \cup \dots \cup (L \setminus e_{ij})_N = L, L \equiv G$ . **c)** If a graph  $G$  that belong to isomorphism class  $\Gamma$  is *reconstructable* by its adjacent structures  $GS^{adj}_n$ , then are so also all other graphs  $G_B, G_C, G_D, \dots$  of isomorphism class  $\Gamma$ . **d)** Thus, Ulam's Conjecture is true, but its set up is *destructive* and it verbatim tracking on the ground of canonically represented, i.e. in the form of sign matrices represented graphs is *senseless*.

**Proposition 4.6.** If structure  $GS$  is *deconstructed* to its adjacent *superstructures*  $GS^{sup}_1, \dots, GS^{sup}_m, \dots, GS^{sup}_N$ , i.e. to its smallest superstructures  $(GS \cup e_{ij})_n$ , then their *intersection*  $\cap (GS \cup e_{ij})_n, n^- \in [1, N^-]$ , *reconstruct* the same structure  $GS$ .

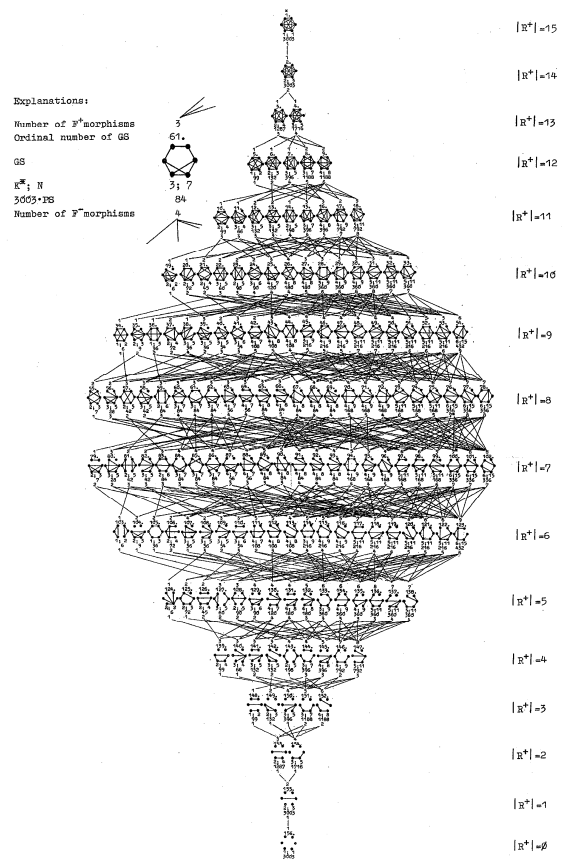
It is possible to set up another kind of reconstruction (Prop. 4.7): Morphism  $F$  is *reversible* – in each adjacent structure  $GS^{adj}$  of  $GS$  exists an “*opposite orbit*”  $\Omega R'$ , whereat used opposite morphism  $F'$  reconstruct the initial structure  $GS, F': GS^{adj} \rightarrow GS$

**Definition 4.3.** A system, that is formed from all the structures with  $|V|$ -elements (i.e. non-isomorphic graphs with  $|V|$  vertices)  $\{GS\}^{|V|}$  and morphisms  $\{F\}$  between these whereby:

- i the set  $\{GS\}^{|V|}$  is decomposed and ordered by numbers of edges  $|E|$  into *structural levels*  $GSL$ ,
- ii where structures  $GS$  in  $GSL$  are ordered by essential structural measures, we call to a **Constructive System of Reconstructions** or *structural changes* denoted by  $CSR^{|V|}$  where  $|V|$  is its *degree* [41].

**Example 4.9.** The lattice of Constructive System of Recognitions  $CSR^{V=6}$  [41] for the structures with  $|V|$  6 elements (see it on the complete text):

So as we know and can be see on the lattice can be various (non-isomorphic) structures have some common adjacent structures. The lattice demonstrates, that edge reconstructivity is true. Non-reconstructable structures cannot exist.



**Structural interpretation the Ulam's Conjecture.** Can be non-isomorphic graphs  $G_A$  and  $G_B$  have structurally equivalent deep sign matrices  $S_A^* \approx S_B^*$ , i.e. the same structure  $GS_A \equiv GS_B$ ?

## **An afterword**

The main sense of 'Mystery' is the principle that graph is an explicate of the structure. Already the classical definitions of structure lead to this fact. On the other words: the structure is under consideration and recognizable on the basis of a graph. Structural attributes are the same as of graphs, but some from theses crop up by semiotic invariants.

If to compare graph theory and structure semiotics then their difference be expressed, as example, on the ground of regularities and isomorphism interpretation. Isomorphism is one-to-one correspondence of vertices with exactness up to their substitutions. On the other hand is isomorphism also one-to-one correspondence of vertices and vertex pairs with exactness up to orbits and some other structural attributes.

On structural aspect are the relations of isomorphism and symmetry treated no solely pairwise, but also as equivalence classes. Such opinions take our approach to reconstruction problem there, where it is. However we call the 'Mystery', it belongs inevitably to graph theory.

Thank you for attention.

Tallinn, June 2010.