The fundamentals of identification and systemization the graphs

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Identification of the graphs means their distinction or recognition in the form of various polynomials, codes, vectors, matrices, etc. It is called also graph canonization [1].

The principal property of the graph is its structure as the attribute of organization the discrete object in the form of relationships between its elements. Structure as such itself is presentable in the form of a graph $G$ where isomorphic graphs have the same structure $GS$.

The main characteristics of structure are its symmetry properties that appear in the form of sets the similar elements (i.e. nodes, node pairs) that on the aspect of group theory to orbits (or equivalence classes, transitivity domains, positions etc.) called.

On the identification and structural properties of graphs was interested B. Weisfeiler in 1976 [2]. After this is the development of this essential direction had weaken, and the interested on this to appear very rarely.

The elementary particles of graph are the binary relations between node pairs. The node-pairs are presented in the adjacency matrix $E$ of a graph and are identifiable by exponentiation the adjacency matrix [3].

1. Identification the structural properties of graphs

The node pairs in initial adjacency matrix $E$ form only two classes: edges and non-edges.

1.1. By multiplication the matrix with itself, $E \times E = E^2$ increases the number $p$ of classes and this continue to a certain degree $E^n$ and then stopped.

1.2. To the identifiers of a class are the elements $e^n_{ij}$ with similar values in the product $E^n$. Essential is, that in the case of maximal-number $p$ determined all the classes of node-pairs.

1.3. The classes of node-pairs coincide with the orbits $O$ of node-pairs in graph $G$. Contraindications not discovered. The orbits of nodes emerge by ordering the product $E^n$.

1.4. The values of identifiers $e^n_{ij}$ be ordered to a sequence $S^n_p = e^n_{i_1} < e^n_{i_2} < \ldots < e^n_p$. For each row $i$ of obtained product $E^n$ attach frequency vector $u^n_i$ that show the number of classes the node pairs in the row.

1.5. Lexicographic reordering of all the rows $i$ and columns $j$ by the frequency vectors $u^n_i$ in $E^n$ form the ordered product $E^n_{ord}$ what represent all the orbits $O$ of node-pairs and nodes. This is the principal invariant of isomorphic graphs.

The fact that the obtained classes coincide with the group-theoretical orbits is remarkable. This fact needs proving or disproving.
Example 1. By exponentiation the adjacency matrix with itself obtained ordered product $E_{ord}^n$ as identifier of isomorphic graphs:

$$E^1 =
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
\end{array}
\begin{array}{cccccc}
i & 0 & 1 & 3 & 3 \\
1 & 2 & 2 & 4 \\
2 & 2 & 4 \\
3 & 3 & 3 \\
4 & 3 & 3 \\
6 & 4 & 4 \\
\end{array}

$$

$$E^2 =
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
3 & 1 & 3 & 0 & 3 & 1 \\
1 & 4 & 1 & 3 & 1 & 3 \\
0 & 3 & 0 & 3 & 0 & 3 \\
3 & 1 & 3 & 0 & 3 & 1 \\
1 & 3 & 1 & 3 & 1 & 4 \\
\end{array}
\begin{array}{cccccc}
i & 0 & 1 & 3 & 4 \\
1 & 2 & 3 & 0 \\
2 & 0 & 3 & 2 & 1 \\
3 & 1 & 2 & 3 & 0 \\
4 & 3 & 0 & 3 & 0 \\
5 & 1 & 2 & 3 & 0 \\
6 & 0 & 3 & 2 & 1 \\
\end{array}

$$

$$E^3 =
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
2 & 10 & 2 & 9 & 2 & 10 \\
10 & 6 & 10 & 3 & 10 & 7 \\
2 & 10 & 2 & 9 & 2 & 10 \\
9 & 3 & 9 & 0 & 9 & 3 \\
2 & 10 & 2 & 9 & 2 & 10 \\
10 & 7 & 10 & 3 & 10 & 6 \\
\end{array}
\begin{array}{cccccc}
i & 0 & 2 & 3 & 6 & 7 & 9 & 10 & k \\
1 & 0 & 3 & 0 & 0 & 0 & 1 & 2 & 2 \\
2 & 0 & 0 & 1 & 1 & 1 & 0 & 3 & 1 \\
3 & 0 & 3 & 0 & 0 & 0 & 1 & 2 & 2 \\
4 & 1 & 0 & 2 & 0 & 0 & 3 & 0 & 3 \\
5 & 0 & 3 & 0 & 0 & 0 & 1 & 2 & 2 \\
6 & 0 & 0 & 1 & 1 & 1 & 0 & 3 & 1 \\
\end{array}

$$

$$E^3_{ord} =
\begin{array}{cccccc}
2 & 6 & 1 & 3 & 5 & 4 \\
6 & 7 & 10 & 10 & 10 & 3 \\
7 & 6 & 10 & 10 & 10 & 3 \\
\end{array}
\begin{array}{cccccc}
i & 0 & 2 & 3 & 6 & 7 & 9 & 10 & k \\
3 & 2 & 0 & 0 & 1 & 1 & 1 & 0 & 3 & 1 \\
6 & 0 & 0 & 1 & 1 & 1 & 0 & 3 & 1 \\
\end{array}

$$

Explanations: There exist three orbits of edges $\{7\}$, $\{9\}$, $\{10\}$, two orbits of “no-edges” $\{2\}$, $\{3\}$ and three orbits of nodes (2, 6), (1, 3, 5), (4).

2. Properties of orbits

2.1. If the nodes $v_a, v_b, \ldots$ of graph $G$ belong to the same orbit $O_k$ then are corresponding subgraphs isomorphic $G_{v_a} \cong G_{v_b} \cong \ldots$, i.e. have the same structure $GS$.

2.2. If the edges $e_1, e_2, \ldots$ of graph $G$ belong to the same orbit $O_n$ then are corresponding greatest subgraphs isomorphic $G_{e_1} \cong G_{e_2} \cong \ldots$, i.e. have the same structure $GS$. 

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2.3. If the non-edges \( ne_1, ne_2, \ldots \) of graph \( G \) belong to the same orbit \( O_n \) then are corresponding smallest super-graphs isomorphic \( G \circ e_1 \cong G \circ e_1 \cong \ldots \), i.e. have the same structure \( GS \).

2.4. Graph that consists of edges of an orbit \( O_n \) called orbit graph \( G_n \).

3. **Properties of orbit-graphs \( G_n \)**

3.1 Orbit-graphs \( G_n \) are natural subgraphs of the graph \( G \). The number of orbit-graphs equal to the number of the orbits of node pairs.

3.2. To each orbit-graph \( G_n \) of the graph \( G \) corresponds an orbit-graph of its complement that coincided.

3.3. Some orbit-graphs \( G_n \) of the graph \( G \) can be appear isomorphic with initial graph (for example, an orbit-graph of the cube is also cube).

3.4. Different orbit-graphs of initial graph \( G \) and/or orbit-graphs of different graphs can be isomorphic or coincides.

3.5. Each orbit-graph has its orbit-graph that we call second degree orbit-graph and so to high degree orbit-graphs.

3.6. A second or high degree orbit-graph can be isomorphic or coincides with a lower degree orbit-graph or initial graph. Coincidence of an orbit-graph and initial graph constitutes a reconstruction of initial structure.

3.7. High degree orbit-graphs no discover more complementary “hidden sides”, these begin to repeat.

3.9. Formation of high degree orbit-graph is a convergent process, it finished with a crop up or reconstruction a low degree or initial graph.

The orbit-graphs discover the hidden sides of initial graph. For example, one orbit-graph of Folkman graph is Petersen graph, etc. These discover also the same attributes of various graphs, For example, Hypercube and Möbius-Kantor graph have some common orbit-graphs. Orbit-graphs are essential means for recognition of structural and symmetric properties of graphs.

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By the orbits of node-pairs \( O_n \) obtained isomorphic greatest subgraphs and isomorphic smallest superstructures called adjacent structures \( GS^{adj}_{n} \).

4. **Properties of adjacent structures \( GS^{adj}_{n} \)**

4.1. All the structures \( GS \) are also adjacent structures \( GS^{adj}_{n} \) of some structures.
4.2. The operation that change structure $GS$ to its adjacent structure $GS^{adj}_n$ called morphism $F_n:GS \rightarrow GS^{adj}_n$.

4.3. Each morphism is reversible – in each adjacent structure $GS^{adj}_n$ in of $GS$ exists a reversible orbit $O^{rev}_n$ where used reversible morphism $F^{rev}_n:GS^{adj}_n \rightarrow GS$ reconstruct the initial structure $GS$.

4.4. The assemblage of sequences of all the adjacent structures with $n$ nodes between empty structure and complete structure form the system of adjacent-structures with $n$ nodes.

Thus, the adjacencies are binary relations between graphs that enable to systematize the non-isomorphic graphs. To this day are compiled only lists of the non-isomorphic graphs [4].

The e adjacencies of obtained can be treated as relationships or morphisms as “edges” between structures. For example: so has the system of structures with 4 nodes 11 elements (structures) and 14 “edges”, with 5 nodes 34 structures and 72 “edges”, with 6 nodes 156 structures and 572 “edges”, etc. All this is connected also with reconstruction conjecture by edges.

Example 2. Upper side of lattice of the system of adjacent-structures with $n=6$ nodes:

Explanation: Such systems are obtainable by sequential working the matrices $E^{ord}_n$. It is detailed, with probability and symmetry characteristics presented completely in work [3].
5. General properties of the systematized adjacent structures $\mathcal{G}^{[1]}$

5.1. If the number of structural levels $m$ in system $\mathcal{G}^{[1]}$ is even number (as in case $|V|=6$ and $|V|=7$), then it lattice is bilaterally symmetric with regard its bisector, which separates the structures $GS$ from their complements $\overline{GS}$.

5.2. If the number of structural levels $m$ in system $\mathcal{G}^{[1]}$ is odd number (as in case $|V|=4$, $|V|=5$, $|V|=8$ and $|V|=9$), then the bisector is a structural level in which be located the structures $GS$, their complements $\overline{GS}$ and also self-complemented structures $GS=\overline{GS}$.

In structural genesis has important role randomness. This is expressed in the form of selection the adjacent structures, i.e. elementary structural changes. The probabilistic characteristics are related with internal diversity of structure, i.e. binary orbits, and have essential meaning in structural research.

6. Probabilistic characteristics of the systems $\mathcal{G}^{[1]}$

6.1. Randomness in the systems $\mathcal{G}$ based on the morphism probabilities $PF_n$.

6.2. There exists transition probability $P_{ij}$ at a structure $GS_i$ to a non-adjacent structure $GS_j$.

6.3. Transition probabilities $P_{ij}$ form the stationary Markov chain $PM$ of structural genesis.

6.4. Existence probability $PS$ of structure $GS$ in the structural level $|R^+|$ of system $\mathcal{G}$ is expressed in the form:

$$PS=\sum_{n=1}^{N}PS_{sup}^{n} \times PF_{sub}^{n}$$

where $n$ is the structural index of binary position, $PS_{sup}^{n}$ existence probability of adjacent superstructure and $PF_{sub}^{n}$ its morphism probability.

6.5. The sum of existence probabilities $PS$ of structures in the structural level $|R^+|$ equal to one, $\sum_{PS}=1$.

6.6. Existence probabilities of structure and its complement are equal, $PS(GS)=PS(\overline{GS})$.

6.7. Distribution of the probabilities $PS$ in the structure levels approach to logarithmic normal distribution.

6.8. Between symmetry values $SR$ and existence probabilities $PS$ exists a strong negative correlation.

The systems of adjacent structures are related with the reconstruction problem that is known as Ulam’s Conjecture that reflects the isomorphism relations between two graphs and their $(G\setminus v_i)$-subgraphs. It is formulated as follows: “If for each $i$, the sub-graphs $G_i=G\setminus v_i$ and $H_i=H\setminus u_i$ are isomorphic, then the graphs $G$ and $H$ are isomorphic.”
This problem has been over the past half century, one of under active consideration graph theoretical problem, but the ultimate solutions have only some graph classes. Why so? On the structural aspect are the attempts of solution the conjecture by its wording senseless, because, if given graphs $G$ and $H$ then on the ground of their structural identifiers $\text{E}^\text{ord}_G$ and $\text{E}^\text{ord}_H$ we obtain the complete information about corresponding graphs, their isomorphism or non-isomorphism and of their adjacent graphs. Other approaches are meaningless for us here.

Evidently be interested on the question: contains the collection of subgraphs $G_{\mathcal{V}_i}$ of $G$ enough information about graph $G$ itself? Ulam’s Conjecture treats the reconstruction on the aspect of removing of the vertices, but we treat it on the aspect of adding and removing of edges. This not changes the essence of reconstruction, because all remains to the frame of graphs (structures) and their adjacent-graphs (adjacent-structures), i.e. in our case to the frame of morphisms $F_n$.

Already old master W. T. Tutte emphasized that reconstruction-problem must be solve on the basis of isomorphism classes, (i.e. structures) that we also have followed [5].

References